Intensity Distribution inside an Isotropically Scattering Semi-infinite Medium

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The intensity distribution inside an absorbing-isotropically scattering, semi-infinite medium exposed to collimated radiation is expressed in terms of two universal functions $a(\tau,\mu)$ and $b(\tau,\mu)$. For incident radiation of the form μ^m, the intensity distribution is expressed in terms of source functions and the $a(\tau,\mu)$ and $b(\tau,\mu)$ functions. Initial value differential equations for these functions are derived and solved numerically. Graphical and tabulated results are presented.

Introduction

INTENSITY distribution in a semi-infinite, absorbing-isotropically scattering medium which is exposed to collimated radiation involves three variables: τ the optical depth, μ the cosine of the polar angle of the intensity, and μ_0 the cosine of the polar angle of incident radiation. In this paper the situation is simplified by expressing the intensity in terms of universal functions, each involving only two variables.

The decomposition of the intensity into universal functions has been accomplished by Kagiwada and Kalaba,[1,2] and Matsumoto[3] for a finite isotropically scattering atmosphere whose upper surface is illuminated by collimated radiation. The intensity is expressed algebraically in terms of the source function and two other functions. Applying invariant imbedding techniques, Kagiwada and Kalaba[1,2] obtained numerical results for these functions. The present investigation considers a semi-infinite medium subjected to various directional loadings, including a polynomial in μ.

Expressions for Intensity

The situation considered in the present investigation is shown in Fig. 1. A semi-infinite, isotropically scattering medium is exposed to radiation, $I^+(0,\mu,\phi)$. The radiant intensity inside the medium is governed by the transport equation, i.e.,

$$\mu\frac{\mathrm{d}I^+}{\mathrm{d}\tau}+I^+=S(\tau) \tag{1a}$$

$$-\mu\frac{\mathrm{d}I^-}{\mathrm{d}\tau}+I^-=S(\tau) \tag{1b}$$

where the source function is given by

$$S(\tau)=\frac{\omega}{4\pi}\int_0^{2\pi}\int_0^1 I^+(0,\mu,\phi)\exp\left(\frac{-\tau}{\mu}\right)\mathrm{d}\mu\mathrm{d}\phi + \frac{\omega}{2}\int_0^\infty S(t)E_1(|\tau-t|)\mathrm{d}t \tag{2}$$

τ is the optical depth, $\tau=\int_0^x\beta\mathrm{d}x'$; ω is the albedo, $\omega=\sigma/\beta$; β is the extinction coefficient, $\beta=\sigma+\kappa$; σ is the scattering coefficient; κ is the absorption coefficient; and $E_1(\tau)$ is the exponential integral of order one. I^+ is the intensity in the positive τ direction, while I^- represents the intensity in the negative τ direction. Emission in the medium is neglected, and the medium is assumed homogeneous. Equations (1) and (2) are for a single frequency; however, for clarity, the frequency subscript is not employed.

If the incident radiation is collimated,

$$I^+(0,\mu,\phi)=I_0\delta(\mu-\mu_0)\delta(\phi-\phi_0) \tag{3}$$

the source function can be expressed as

$$S(\tau)=\omega I_0 J(\tau,\mu_0)/4\pi \tag{4}$$

where

$$J(\tau,\mu_0)=\exp\left(\frac{-\tau}{\mu_0}\right)+\frac{\omega}{2}\int_0^\infty J(t,\mu_0)E_1(|\tau-t|)\mathrm{d}t \tag{5}$$

Note that δ is the Dirac delta function. The source function $J(\tau,\mu_0)$ satisfies the following differential equation:[4]

$$\frac{\partial J(\tau,\mu)}{\partial\tau}=\frac{-J(\tau,\mu)}{\mu}+H(\mu)\Phi(\tau) \tag{6}$$

with initial condition $J(0,\mu)=H(\mu)$. Chandrasekhar's H function is governed by a nonlinear integral equation[5]

$$\frac{1}{H(\mu)}=\sqrt{1-\omega}+\frac{\omega}{2}\int_0^1\frac{\mu' H(\mu')}{\mu+\mu}\mathrm{d}\mu' \tag{7}$$

The function $\Phi(\tau)$ satisfies the following integral equation:

$$\Phi(\tau)=\frac{\omega}{2}E_1(\tau)+\frac{\omega}{2}\int_0^\infty\Phi(t)E_1(|\tau-t|)\mathrm{d}t \tag{8}$$

and is related to $J(\tau,\mu)$ by

$$\Phi(\tau)=\frac{\omega}{2}\int_0^1 J(\tau,\mu')\frac{\mathrm{d}\mu'}{\mu'} \tag{9}$$

Combination of Eqs. (6) and (9) yields

$$\frac{\partial J(\tau,\mu)}{\partial\tau}=-\frac{J(\tau,\mu)}{\mu}+\frac{\omega}{2}H(\mu)\int_0^1 J(\tau,\mu')\frac{\mathrm{d}\mu'}{\mu'} \tag{10}$$

The transport equations (1) can be easily solved to yield:

$$I^+(\tau,\mu,\phi;\ \mu_0,\phi_0)=I_0\delta(\mu-\mu_0)\delta(\phi-\phi_0)\exp(-\tau/\mu) + \frac{\omega I_0}{4\pi}\int_0^\tau J(t,\mu_0)\exp\left[\frac{-(\tau-t)}{\mu}\right]\frac{\mathrm{d}t}{\mu} \tag{11}$$

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and

$$I^-(\tau,\mu;\mu_0)=\frac{\omega I_0}{4\pi}\int_\tau^\infty J(t,\mu_0)\exp\left[\frac{-(t-\tau)}{\mu}\right]\frac{dt}{\mu} \tag{12}$$

I^+ can be thought of as being composed of two components: I_δ^+ accounts for single scattering of the incident radiation, while I_0^+ accounts for multiple scattering. In other words

$$I^+(\tau,\mu,\phi;\ \mu_0,\phi_0)=I_\delta^+(\tau,\mu,\phi;\ \mu_0,\phi_0)+I_0^+(\tau,\mu;\ \mu_0) \tag{13}$$

where

$$I_\delta^+(\tau,\mu,\phi;\ \mu_0,\phi_0)=I_0\delta(\mu-\mu_0)\delta(\phi-\phi_0)\exp(-\tau/\mu) \tag{14}$$

and

$$I_0^+(\tau,\mu;\ \mu_0)=\frac{\omega I_0}{4\pi}\int_0^\tau J(t,\mu_0)\exp\left[\frac{-(\tau-t)}{\mu}\right]\frac{dt}{\mu} \tag{15}$$

Thus, for a given albedo, three variables (τ,μ, and μ_0) are required to tabulate the intensities, I_0^+ and I^-.

Starting from Eq. (15), it is possible to express I_0^+ in terms of the a and J functions. Letting $u=J(t,\mu_0)$ and $dv=\exp[(\tau-t)/\mu]dt/\mu$, integration of Eq. (12) by parts results in

$$I_0^+(\tau,\mu;\ \mu_0)=\frac{\omega I_0}{4\pi}\{J(\tau,\mu_0)-J(0,\mu_0)\exp\left(\frac{-\tau}{\mu}\right)$$
$$-\int_0^\tau\frac{\partial J(t,\mu_0)}{\partial t}\exp\left[\frac{-(\tau-t)}{\mu}\right]dt\} \tag{16}$$

Substitution of Eq. (6) into Eq. (16) yields

$$I_0^+(\tau,\mu;\ \mu_0)=\frac{\omega I_0}{4\pi}\{J(\tau,\mu_0)-H(\mu_0)(\exp\left(\frac{-\tau}{\mu}\right)$$
$$+\frac{1}{\mu_0}\int_0^\tau J(t,\mu_0)\exp\left[\frac{-(\tau-t)}{\mu}\right]dt$$
$$-H(\mu_0)\int_0^\tau\Phi(t)\exp\left[\frac{-(\tau-t)}{\mu}\right]dt\} \tag{17}$$

Utilization of Eq. (15) and rearrangement of Eq. (17) gives

$$I_0^+(\tau,\mu;\ \mu_0)=\frac{\omega I_0\mu_0}{4\pi(\mu-\mu_0)}\{-J(\tau,\mu_0)+H(\mu_0)\exp\left(\frac{-\tau}{\mu}\right)$$
$$+H(\mu_0)\int_0^\tau\Phi(t)\exp\left[\frac{-(\tau-t)}{\mu}\right]dt\} \tag{18}$$

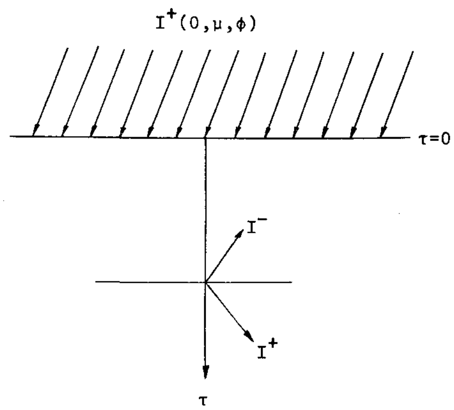

Fig. 1 Physical situation.

Defining the a function as follows

$$a(\tau,\mu)=\exp\left(\frac{-\tau}{\mu}\right)+\int_0^\tau\Phi(t)\exp\left[\frac{-(\tau-t)}{\mu}\right]dt \tag{19}$$

the intensity in the positive τ direction can be expressed as

$$I_0^+(\tau,\mu;\ \mu_0)=\frac{\omega I_0\mu_0}{4\pi(\mu-\mu_0)}[-J(\tau,\mu_0)+H(\mu_0)a(\tau,\mu)] \tag{20}$$

Using a similar procedure, I^- can be expressed in terms of the functions $J(\tau,\mu)$ and $b(\tau,\mu)$. Letting $u=J(t,\mu_0)$ and $dv=\exp[-(t-\tau)/\mu]dt/\mu$ in Eq. (12) and integrating by parts results in

$$I^-(\tau,\mu;\ \mu_0)=\frac{\omega I_0}{4\pi}J(\tau,\mu_0)$$
$$+\frac{\omega I_0}{4\pi}\int_\tau^\infty\frac{\partial J(t,\mu_0)}{\partial t}\exp\left[\frac{-(t-\tau)}{\mu}\right]dt \tag{21}$$

Substitution of Eq. (6) into Eq. (21) yields

$$I^-(\tau,\mu;\ \mu_0)=\frac{\omega I_0}{4\pi}\{J(\tau,\mu_0)$$
$$-\frac{1}{\mu_0}\int_\tau^\infty J(t,\mu_0)\exp\left[\frac{-(t-\tau)}{\mu}\right]dt$$
$$+H(\mu_0)\int_\tau^\infty\Phi(t)\exp\left[\frac{-(t-\tau)}{\mu}\right]dt\} \tag{22}$$

Utilization of Eq. (9) and rearrangement of Eq. (21) gives

$$I^-(\tau,\mu;\ \mu_0)=\frac{\omega I_0\mu_0}{4\pi(\mu+\mu_0)}[J(\tau,\mu_0)+H(\mu_0)b(\tau,\mu)] \tag{23}$$

where the b function is defined as

$$b(\tau,\mu)=\int_\tau^\infty\Phi(t)\exp\left[\frac{-(t-\tau)}{\mu}\right]dt \tag{24}$$

Alternate expressions for the a and b functions can be obtained by differentiating Eqs. (19) and (24) with respect to τ, i.e.,

$$\frac{\partial a(\tau,\mu)}{\partial\tau}=\frac{-\exp(-\tau/\mu)}{\mu}-\frac{1}{\mu}\int_0^\tau\Phi(t)$$
$$\times\exp\left[\frac{-(\tau-t)}{\mu}\right]dt+\Phi(\tau) \tag{25}$$

$$\frac{\partial b(\tau,\mu)}{\partial\tau}=\frac{1}{\mu}\int_\tau^\infty\Phi(t)\exp\left[\frac{-(t-\tau)}{\mu}\right]dt-\Phi(\tau) \tag{26}$$

Using the definitions of $a(\tau,\mu)$ and $b(\tau,\mu)$ reduces these equations to

$$\frac{\partial a(\tau,\mu)}{\partial\tau}=-\frac{a(\tau,\mu)}{\mu}+\Phi(\tau) \tag{27}$$

$$\frac{\partial b(\tau,\mu)}{\partial\tau}=\frac{b(\tau,\mu)}{\mu}-\Phi(\tau) \tag{28}$$

Initial conditions for these two equations are obtained by evaluating Eqs. (19) and (23) at $\tau=0$ and are

$$a(0,\mu)=1 \tag{29}$$

$$I^-(0,\mu;\mu_0)=\frac{\omega I_0\mu_0}{4\pi(\mu+\mu_0)}H(\mu_0)[1+b(0,\mu)] \tag{30}$$

From Chandrasekhar

$$I^-(0,\mu;\mu_0)=\frac{\omega I_0}{4\pi}\int_0^\infty J(t,\mu_0)\exp\left(\frac{-t}{\mu}\right)\frac{dt}{\mu}$$

$$=\frac{\omega I_0\mu_0}{4\pi(\mu+\mu_0)}H(\mu_0)H(\mu) \tag{31}$$

Thus, the initial condition for the b function is

$$b(0,\mu)=H(\mu)-1 \tag{32}$$

Multiplication of Eqs. (27) and (29) by $H(\mu)$ and comparison with Eq. (6) and its initial condition reveals

$$a(\tau,\mu)=J(\tau,\mu)/H(\mu) \tag{33}$$

This relationship eliminates the need for solving both Eqs. (6) and (27) and reduces Eqs. (20) and (23):

$$I_0^+(\tau,\mu;\mu_0)=\frac{\omega I_0\mu_0 H(\mu_0)}{4\pi(\mu-\mu_0)}[a(\tau,\mu)-a(\tau,\mu_0)] \tag{34}$$

$$I^-(\tau,\mu;\mu_0)=\frac{\omega I_0\mu_0 H(\mu_0)}{4\pi(\mu+\mu_0)}[a(\tau,\mu_0)+b(\tau,\mu)] \tag{35}$$

When $\mu=\mu_0$, Eq. (34) is undeterminate, and an alternate one is needed. Evaluation of Eq. (15) at $\mu=\mu_0$ yields

$$I_0^+(\tau,\mu_0;\mu_0)=\omega I_0 H(\mu_0)c(\tau,\mu_0)/4\pi\mu_0 \tag{36}$$

where

$$c(\tau,\mu_0)=\int_0^\tau a(t,\mu_0)\exp\left[\frac{-(\tau-t)}{\mu_0}\right]dt \tag{37}$$

Differentiation of this equation with respect to τ gives

$$\frac{\partial c(\tau,\mu_0)}{\partial\tau}=\frac{-c(\tau,\mu_0)}{\mu_0}+a(\tau,\mu_0) \tag{38}$$

with initial condition $c(0,\mu_0)=0$.

Diffuse Incident Radiation

If the incident radiation is diffuse, $I^+(0,\mu,\phi)=I_d$, the source function can be expressed as

$$S(\tau)=\omega I_d\phi(\tau) \tag{39}$$

where

$$\phi(\tau)=\frac{1}{2}E_2(\tau)+\frac{\omega}{2}\int_0^\infty\phi(t)E_1(|\tau-t|)dt \tag{40}$$

The diffuse source function is simply related to the collimated source function as follows

$$\phi(\tau)=\frac{1}{2}\int_0^1 J(\tau,\mu)d\mu \tag{41}$$

The differential equation for this function can be found from Eq. (6) and is given by

$$\frac{\partial\phi(\tau)}{\partial\tau}=[\omega\phi(0)-1]\Phi(\tau)/\omega=-\sqrt{1-\omega}\,\Phi(\tau)/\omega \tag{42}$$

with initial condition $\phi(0)=\frac{1}{2}\int_0^1 H(\mu)d\mu=[1-\sqrt{1-\omega}]/\omega$. Following a procedure similar to the collimated situation yields

$$I^+(\tau,\mu)=I_d\exp\left(\frac{-\tau}{\mu}\right)+\omega I_d\{\phi(\tau)-\phi(0)\exp\left(\frac{-\tau}{\mu}\right)$$

$$-\int_0^\tau\frac{\partial\phi}{\partial t}\exp\left[\frac{-(\tau-t)}{\mu}\right]dt\} \tag{43}$$

and

$$I^-(\tau,\mu)=\omega I_d\{\phi(\tau)-\frac{\sqrt{1-\omega}}{\omega}\int_\tau^\infty\Phi(t)\exp\left[\frac{-(t-\tau)}{\mu}\right]dt\} \tag{44}$$

Substitution of Eq. (42) into these equations gives

$$I^+(\tau,\mu)=I_d\exp\left(\frac{-\tau}{\mu}\right)+\omega I_d\{\phi(\tau)-\phi(0)\exp\left(\frac{-\tau}{\mu}\right)$$

$$+\frac{\sqrt{1-\omega}}{\omega}\int_0^\tau\Phi(t)\exp\left[\frac{-(\tau-t)}{\mu}\right]dt\} \tag{45}$$

and

$$I^-(\tau,\mu)=\omega I_d\{\phi(\tau)-\frac{\sqrt{1-\omega}}{\omega}\int_\tau^\infty\Phi(t)\exp\left[\frac{-(t-\tau)}{\mu}\right] \tag{46}$$

Utilization of Eqs. (19) and (24) produces

$$I^+(\tau,\mu)=I_d[\omega\phi(\tau)+\sqrt{1-\omega}\,a(\tau,\mu)] \tag{47}$$

$$I^-(\tau,\mu)=I_d[\omega\phi(\tau)-\sqrt{1-\omega}\,b(\tau,\mu)] \tag{48}$$

Inspection of Eqs. (47) and (48) reveals that the intensity is independent of direction for pure scattering ($\omega=1$).

Incident Radiation of the Form $I_m\mu^m$

If the radiation incident on the medium is proportional to the cosine of the polar angle raised to the mth power, i.e.,

$$I^+(0,\mu,\phi)=I_m\mu^m \tag{49}$$

the source function can be expressed as

$$S(\tau)=\omega I_m s_m(\tau) \tag{50}$$

where $s_m(\tau)$ is defined by

$$s_m(\tau)=\frac{1}{2}E_{m+2}(\tau)+\frac{\omega}{2}\int_0^\infty s_m(t)E_1(|\tau-t|)dt \tag{51}$$

This dimensionless source function is simply related to the collimated source function as follows:

$$s_m(\tau)=\frac{1}{2}\int_0^1 J(\tau,\mu)\mu^m d\mu \tag{52}$$

Multiplication of Eq. (6) by $\mu^m d\mu/2$ and integration from 0 to 1 yields

$$\frac{ds_m(\tau)}{d\tau}=-s_{m-1}(\tau)+s_m(0)\Phi(\tau) \tag{53}$$

with initial condition, $s_m(0)=\frac{1}{2}\int_0^1 H(\mu)\mu^m d\mu$. Following a procedure similar to the collimated case gives

$$I_m^+(\tau,\mu)=I_m i_m^+(\tau,\mu)=I_m\left\{\mu^m\exp\left(\frac{-\tau}{\mu}\right)+\omega s_m(\tau)-\omega s_m(0)\right.$$

$$\left.\times\exp\left(\frac{-\tau}{\mu}\right)-\omega\int_0^\tau s_m'(t)\exp\left[\frac{-(\tau-t)}{\mu}\right]dt\right\} \tag{54}$$

and

$$I_m^-(\tau,\mu)=I_m i_m^-(\tau,\mu)=I_m\left\{\omega s_m(\tau)\right.$$

$$\left.+\omega\int_\tau^\infty s_m'(t)\exp\left[\frac{-(t-\tau)}{\mu}\right]dt\right\} \tag{55}$$

Substitution of Eq. (53) into these equations yields

$$i_m^+(\tau,\mu)=\omega s_m(\tau)+\mu^m \exp\left(\frac{-\tau}{\mu}\right)$$
$$-\omega\int_0^\tau s_{m-1}(t)\exp\left[\frac{-(\tau-t)}{\mu}\right]\mathrm{d}t$$
$$-\omega s_m(0)\left\{\exp\left(\frac{-\tau}{\mu}\right)+\int_0^\tau \Phi(t)\exp\left[\frac{-(\tau-t)}{\mu}\right]\mathrm{d}t\right\} \quad (56)$$

and

$$i_m^-(\tau,\mu)=\omega s_m(\tau)-\omega\int_\tau^\infty s_m(t)\exp\left[\frac{-(\tau-t)}{\mu}\right]\mathrm{d}t$$
$$+s_m(0)\int_\tau^\infty \Phi(t)\exp\left[\frac{-(t-\tau)}{\mu}\right]\mathrm{d}t \quad (57)$$

Utilization of Eqs. (19) and (24) produces the following recurrence formulas:

$$i_m^+(\tau,\mu)=\omega s_m(\tau)+\mu i_{m-1}^+(\tau)-\omega s_m(0)a(\tau,\mu) \quad (58)$$

$$i_m^-(\tau,\mu)=\omega s_m(\tau)-\mu i_{m-1}^-(\tau)-\omega s_m(0)b(\tau,\mu) \quad (59)$$

Since $m=0$ corresponds to the diffuse case,

$$i_0^+(\tau,\mu)=\omega\phi(\tau)+\sqrt{1-\omega}\;a(\tau,\mu) \quad (60)$$

$$i_0^-(\tau,\mu)=\omega\phi(\tau)-\sqrt{1-\omega}\;b(\tau,\mu) \quad (61)$$

The result for $m=1$ follows immediately:

$$i_1^+(\tau,\mu)=\omega\; q(\tau)+\mu[\omega\phi(\tau)+\sqrt{1-\omega}\;a(\tau,\mu)]-\omega\; q(0)\,a(\tau,\mu) \quad (62)$$

$$i_1^-(\tau,\mu)=\omega q(\tau)-\mu[\omega\phi(\tau)-\sqrt{1-\omega}\;b(\tau,\mu)]-\omega\; q(0)\; b(\tau,\mu) \quad (63)$$

where $q(\tau)=s_1(\tau)$ is Hopf's function. Inspection of Eqs. (58) and (59) reveals that the directional character of the intensity distribution is found in the a and b functions.

For a polynomial representation of the incident intensity, i.e.,

$$I^+(0,\mu,\phi)=\sum_{m=0}^{M} I_m \mu^m \quad (64)$$

the source function and intensity distribution are found by superposition:

$$S(\tau)=\sum_{m=0}^{M} I_m s_m(\tau) \quad (65)$$

$$I_m^+(\tau,\mu)=\sum_{m=0}^{M} I_m i_m^+(\tau,\mu) \quad (66)$$

$$I^-(\tau,\mu)=\sum_{m=0}^{M} I_m i_m^-(\tau,\mu) \quad (67)$$

Physical Interpretation for $a(\tau,\mu)$ and $b(\tau,\mu)$

If the incident radiation is proportional to the secant of the polar angle, i.e.,

$$I^+(0,\mu,\phi)=I_A/\mu \quad (68)$$

the source function can be expressed as

$$S(\tau)=I_A\Phi(\tau) \quad (69)$$

where $\Phi(\tau)$ is defined by the integral equation (8). The transport equation, Eq. (1) for this case is

$$\mu\frac{\mathrm{d}I^+}{\mathrm{d}\tau}+I^+=I_A\Phi(\tau) \quad (70)$$

$$-\mu\frac{\mathrm{d}I^-}{\mathrm{d}\tau}+I^-=I_A\Phi(\tau) \quad (71)$$

Comparison of these equations with Eqs. (27) and (28) yields

$$I^+(\tau,\mu)=I_A a(\tau,\mu)/\mu \quad (72)$$

$$I^-(\tau,\mu)=I_A b(\tau,\mu)/\mu \quad (73)$$

Thus, $a(\tau,\mu)$ and $b(\tau,\mu)$ are essentially the intensities in a semi-infinite, isotropically scattering medium subjected to the boundary condition in Eq. (68).

An alternate physical interpretation for $\Phi(\tau)$ is possible. For an emitting medium with no incident radiation, the transport equation becomes

$$\mu\frac{\mathrm{d}I}{\mathrm{d}\tau}+I=(1-\omega)I_b(\tau)+S(\tau) \quad (74)$$

where the source function satisfies the following equation

$$S(\tau)=\frac{\omega}{2}\int_0^\infty [(1-\omega)I_b(t)+S(t)]E_1(|\tau-t|)\mathrm{d}t \quad (75)$$

If emission is restricted to the upper boundary, i.e.,

$$I_b(\tau)=I_B\delta(t) \quad (76)$$

Eq. (75) becomes

$$S(\tau)=\frac{\omega}{2}I_B(1-\omega)E_1(\tau)+\frac{\omega}{2}\int_0^\infty S(t)E_1(|\tau-t|)\mathrm{d}t \quad (77)$$

Thus, the source function can be expressed in terms of $\Phi(\tau)$

$$S(\tau)=I_B(1-\omega)\Phi(\tau) \quad (78)$$

Numerical Procedure

In this section, the numerical procedure used to calculate $a(\tau,\mu)$, $b(\tau,\mu)$, $c(\tau,\mu)$, and $\phi(\tau)$ is presented. The key step in this procedure is the calculation of $\Phi(\tau)$. The integral in Eq. (9) is broken into two parts, i.e.,

$$\Phi(\tau)=\frac{\omega}{2}\int_0^\epsilon J(\tau,\mu')\frac{\mathrm{d}\mu'}{\mu'}+\frac{\omega}{2}\int_\epsilon^1 J(\tau,\mu')\frac{\mathrm{d}\mu'}{\mu'} \quad (79)$$

Using Gaussian quadrature of order n_1 for the first and n_2 for the second, the function $\Phi(\tau)$ can be represented by

$$\Phi(\tau)=\frac{\omega}{2}\sum_{k=1}^{N} w_k J_k(\tau)/\mu_k=\frac{\omega}{2}\sum_{k=1}^{N} w_k H(\mu_k)a_k(\tau)/\mu_k \quad (80)$$

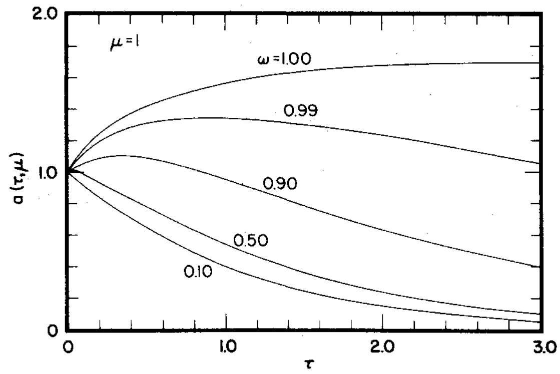

Fig. 2 Influence of albedo on $a(\tau,\mu)$ for $\mu=1$.

where w_k and μ_k are the appropriate Gaussian weights and abscissas; $N = n_1 + n_2$, $J_k(\tau) = J(\tau,\mu_k)$, and $a_k(\tau) = a(\tau,\mu_k)$. Evaluating Eq. (6) at the abscissas

$$\frac{\mathrm{d}a_i(\tau)}{\mathrm{d}\tau} = \frac{-a_i(\tau)}{\mu_i} + H(\mu_i)\Phi(\tau) \qquad i=1,2...N \qquad (81)$$

produces a system of N simultaneous differential equations. Equations (81) are coupled to each other by Eq. (80). Initial conditions, $H(\mu_i)$, are determined by solving Eq. (7) by successive approximations. In order to obtain $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ at even values of μ, additional differential equations must be solved. Evaluating Eqs. (27, 28, and 38) at $\mu_{N+1}, \mu_{N+2}, \ldots \mu_{N+M}$ yields

$$\frac{\mathrm{d}a_i(\tau)}{\mathrm{d}\tau} = -\frac{a_i(\tau)}{\mu_i} + \Phi(\tau)$$

$$\frac{\mathrm{d}b_i(\tau)}{\mathrm{d}\tau} = \frac{b_i(\tau)}{\mu_i} - \Phi(\tau) \qquad i=N+1,\ N+2,\ldots N+M$$

$$\frac{\mathrm{d}c_i(\tau)}{\mathrm{d}\tau} = -\frac{c_i(\tau)}{\mu_i} + a_i(\tau) \qquad (82)$$

with initial conditions determined from Eq. (7). Equations (81) and (82) plus Eq. (42) represent a system of $N+3M+1$ ordinary differential equations. These equations were solved numerically with a fourth-order Runge-Kutta method.

The differential equation for $b(\tau,\mu)$ is ill-conditioned. A small error in the initial condition $b(0,\mu)$ produces a large error in $b(\tau,\mu)$ when τ/μ is large. This behavior can be seen by considering the following differential equation

$$\frac{\partial B(\tau,\mu)}{\partial \tau} = \frac{B(\tau,\mu)}{\mu} - \Phi(\tau) \qquad (83)$$

with

$$B(0,\mu) = b(0,\mu) + \epsilon(0,\mu) \qquad (84)$$

where $\epsilon(0,\mu)$ is small. Letting $B(\tau,\mu) = b(\tau,\mu) + \epsilon(\tau,\mu)$, and using Eq. (28), we obtain a differential equation for $\epsilon(\tau,\mu)$

$$\frac{\partial \epsilon(\tau,\mu)}{\partial \tau} = \frac{\epsilon(\tau,\mu)}{\mu} \qquad (85)$$

which is easily solved to yield

$$\epsilon(\tau,\mu) = \epsilon(0,\mu)\exp(\tau/\mu) \qquad (86)$$

An error of $(10)^{-10}$ in $b(0,\mu)$ would yield an error of $2.20(10)^{-6}$ in $b(\tau,\mu)$ when $\tau/\mu = 10$, an error of $4.85(10)^{-2}$ when $\tau/\mu = 20$, and an error of $1.07(10)^3$ when $\tau/\mu = 30$.

Thus, when τ/μ is large, an alternate computational procedure for finding $b(\tau,\mu)$ is required. With the transformation, $z = (t-\tau)/\mu$, Eq. (24) becomes

$$b(\tau,\mu) = \mu\int_0^\infty \Phi(\tau+\mu z)e^{-z}\mathrm{d}z \qquad (87)$$

This equation can be evaluated numerically using Laguerre quadrature once $\Phi(\tau)$ is determined from Eq. (80).

Results

Numerical results for $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ are presented in Tables 1-5 for $\omega = 1.00$, 0.99, 0.5, and 0.1, respectively. These results were obtained with $\epsilon = 0.2$, $n_1 = 25$, and $n_2 = 25$. However, accurate results can be obtained without dividing the interval and without the large number of quadrature points.

The influence of albedo on $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ for $\mu = 1$ is illustrated in Figs. 2-4, respectively. When absorption is increased, the albedo is decreased. A small amount of absorption causes a large change in these functions. This behavior is most pronounced at large optical depths. For pure scattering ($\omega = 1$), the functions $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ becomes constant at the large depths; however, with absorption ($\omega < 1$) these functions approach zero. All three functions decrease as absorption increases.

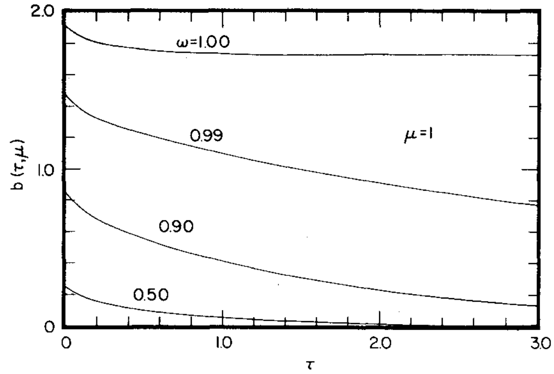

Fig. 3 Influence of albedo on $b(\tau,\mu)$ for $\mu = 1$.

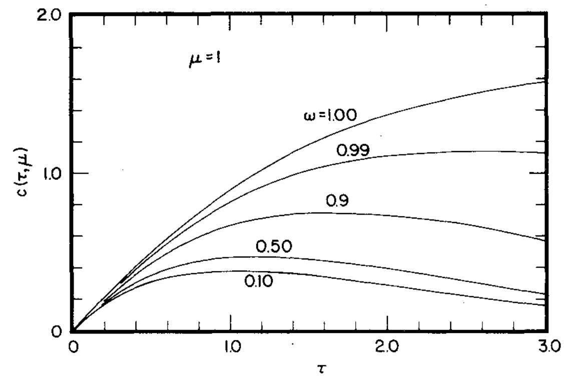

Fig. 4 Influence of albedo on $c(\tau,\mu)$ for $\mu = 1$.

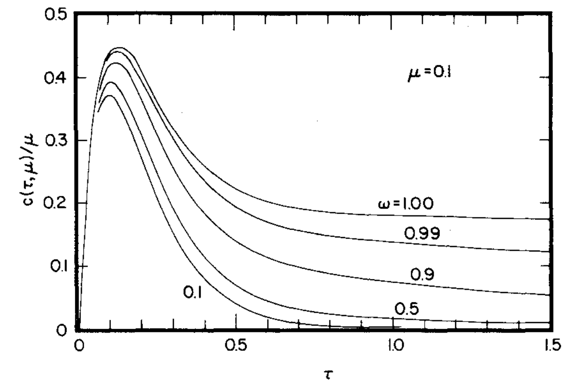

Fig. 5 Influence of albedo on $c(\tau,\mu)$ for $\mu = 0.1$.

While the $b(\tau,\mu)$ function decreases with optical depth (τ) for any μ or ω, the behavior of the $a(\tau,\mu)$ and $c(\tau,\mu)$ functions is more complicated. The $a(\tau,\mu)$ and $c(\tau,\mu)$ functions first increase with τ, reach a maximum, and then decrease. The maximum occurs at a smaller τ as ω or μ is decreased. This behavior is most pronounced in the $c(\tau,\mu)$ function as illustrated in Figs. 4 and 5. However, for pure scattering ($\omega = 1$) and μ near unity, the $a(\tau,\mu)$ function only increases with τ.

Numerical results for $\Phi(\tau)$, $\phi(\tau)$, and $q(\tau)$ are presented in Tables 6 and 7 for $\omega = 1.00$, 0.99, 0.9, 0.5, and 0.1. The influence of absorption on these functions is similar to that for the $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ functions.

Table 1 Functions $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ for $\omega = 1.00$

τ	a(τ,μ) μ=1.0	a(τ,μ) μ=0.5	a(τ,μ) μ=0.1	b(τ,μ) μ=1.0	b(τ,μ) μ=0.5	b(τ,μ) μ=0.1	c(τ,μ) μ=1.0	c(τ,μ) μ=0.5	c(τ,μ) μ=0.1
0	1.00000	1.00000	1.00000	1.90781	1.01278	0.24735	0.000000	0.000000	0.000000
0.001	1.00375	1.00275	0.99478	1.90497	1.01005	0.24506	0.001001	0.001000	0.000993
0.002	1.00679	1.00478	0.98890	1.90283	1.00803	0.24346	0.002005	0.002001	0.001970
0.005	1.01471	1.00969	0.97046	1.89758	1.00311	0.23975	0.005027	0.005002	0.004807
0.010	1.02603	1.01599	0.93915	1.89062	0.99667	0.23519	0.01009	0.009992	0.009228
0.020	1.04539	1.02529	0.87803	1.87975	0.98679	0.22865	0.02030	0.01990	0.016989
0.050	1.09187	1.04179	0.71811	1.85720	0.96677	0.21669	0.05132	0.04887	0.033087
0.10	1.15204	1.05326	0.52671	1.83309	0.94600	0.20570	0.10360	0.09412	0.043974
0.20	1.24274	1.05298	0.32594	1.80414	0.92187	0.19436	0.20799	0.17265	0.041174
0.30	1.31200	1.04064	0.24405	1.78647	0.90758	0.18827	0.30994	0.23626	0.032372
0.40	1.36796	1.02418	0.20964	1.77444	0.89803	0.18446	0.40809	0.28700	0.025924
0.50	1.41445	1.00676	0.19431	1.76574	0.89124	0.18187	0.50175	0.32699	0.022165
0.60	1.45373	0.98986	0.18688	1.75923	0.88622	0.18001	0.59055	0.35816	0.020138
0.80	1.51623	0.95991	0.18044	1.75031	0.87943	0.17760	0.75315	0.40054	0.018477
1.0	1.56316	0.93600	0.17770	1.74469	0.87522	0.17616	0.89607	0.42454	0.017935
1.5	1.63826	0.89846	0.17492	1.73752	0.86994	0.17442	1.17609	0.44441	0.017531
2.0	1.67884	0.88081	0.17395	1.73460	0.86783	0.17375	1.36737	0.44386	0.017410
3.0	1.71431	0.86914	0.17338	1.73267	0.86646	0.17333	1.57930	0.43699	0.017340
4.0	1.72598	0.88673	0.17325	1.73222	0.86614	0.17324	1.66957	0.43410	0.017326
5.0	1.72994	0.86620	0.17322	1.73210	0.86606	0.17321	1.70690	0.43329	0.017322
6.0	1.73131	0.86607	0.17321	1.73207	0.86604	0.17321	1.72204	0.43308	0.017321
8.0	1.73196	0.86603	0.17321	1.73205	0.86603	0.17321	1.73050	0.43302	0.017321
10.0	1.73204	0.86603	0.17321	1.73205	0.86603	0.17321	1.73182	0.43301	0.017321

Table 2 Functions $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ for $\omega = 0.99$

τ	a(τ,μ) μ=1.0	a(τ,μ) μ=0.5	a(τ,μ) μ=0.1	b(τ,μ) μ=1.0	b(τ,μ) μ=0.5	b(τ,μ) μ=0.1	c(τ,μ) μ=1.0	c(τ,μ) μ=0.5	c(τ,μ) μ=0.1
0	1.00000	1.00000	1.00000	1.47279	0.84860	0.22488	0.000000	0.000000	0.000000
0.001	1.00354	1.00254	0.99457	1.46972	0.84575	0.22257	0.001001	0.001000	0.000992
0.002	1.00638	1.00437	0.98849	1.46735	0.84360	0.22095	0.002005	0.002001	0.001969
0.005	1.01370	1.00869	0.96948	1.46139	0.83829	0.21716	0.005025	0.005000	0.004804
0.010	1.02404	1.01401	0.93725	1.45324	0.83120	0.21246	0.01008	0.009982	0.009219
0.020	1.04144	1.02138	0.87442	1.43997	0.81998	0.20562	0.02026	0.01986	0.016954
0.050	1.08200	1.03216	0.71014	1.41025	0.79596	0.19274	0.05107	0.04863	0.032905
0.10	1.13207	1.03421	0.51331	1.37435	0.76863	0.18023	0.10265	0.09322	0.043428
0.20	1.20176	1.01556	0.30566	1.32240	0.73179	0.16602	0.20421	0.16932	0.039854
0.30	1.24923	0.98550	0.21938	1.28243	0.70523	0.15720	0.30157	0.22931	0.030436
0.40	1.28281	0.95201	0.18161	1.24866	0.68378	0.15077	0.39347	0.27551	0.023525
0.50	1.30652	0.91824	0.16336	1.21876	0.66541	0.14564	0.47931	0.31027	0.019402
0.60	1.32274	0.88562	0.15323	1.19152	0.64910	0.14132	0.55887	0.33571	0.017065
0.80	1.33867	0.82596	0.14176	1.14242	0.62048	0.13418	0.69911	0.36566	0.014864
1.0	1.33891	0.77437	0.13428	1.09820	0.59533	0.12823	0.81526	0.37662	0.013831
1.5	1.29919	0.67452	0.12057	1.00068	0.54115	0.11599	1.01425	0.36401	0.012308
2.0	1.23014	0.60196	0.10969	0.91522	0.49443	0.10578	1.11223	0.33325	0.011177
3.0	1.06696	0.49612	0.09180	0.76849	0.41487	0.08864	1.12724	0.27413	0.009344
4.0	0.90834	0.41534	0.07714	0.64631	0.34885	0.07451	1.02999	0.22808	0.007850
5.0	0.76795	0.34906	0.06489	0.54380	0.29350	0.06268	0.90040	0.19121	0.006603
6.0	0.64750	0.29364	0.05460	0.45760	0.24697	0.05275	0.77136	0.16073	0.005556
8.0	0.45907	0.20792	0.03866	0.32407	0.17490	0.03735	0.55303	0.11378	0.003934
10.0	0.32518	0.14725	0.02738	0.22950	0.12387	0.02645	0.39271	0.08057	0.002786

Table 3 Functions $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ for $\omega = 0.9$

τ	a(τ,μ) μ=1.0	a(τ,μ) μ=0.5	a(τ,μ) μ=0.1	b(τ,μ) μ=1.0	b(τ,μ) μ=0.5	b(τ,μ) μ=0.1	c(τ,μ) μ=1.0	c(τ,μ) μ=0.5	c(τ,μ) μ=0.1
0	1.00000	1.00000	1.00000	0.85010	0.55603	0.17214	0.000000	0.000000	0.000000
0.001	1.00287	1.00187	0.99390	0.84708	0.55327	0.16998	0.001001	0.001000	0.000992
0.002	1.00509	1.00309	0.98722	0.84470	0.55115	0.16845	0.002004	0.002000	0.001968
0.005	1.01068	1.00568	0.96653	0.83861	0.54583	0.16483	0.005017	0.004992	0.004797
0.010	1.01829	1.00829	0.93175	0.83010	0.53857	0.16027	0.01005	0.009953	0.009191
0.020	1.03047	1.01052	0.86441	0.81590	0.52679	0.15351	0.02015	0.01975	0.016854
0.050	1.05623	1.00703	0.68946	0.78277	0.50047	0.14034	0.05042	0.04800	0.032419
0.10	1.08274	0.98727	0.48075	0.74076	0.46881	0.12682	0.10018	0.09091	0.042048
0.20	1.10740	0.92974	0.26054	0.67711	0.42337	0.11027	0.19509	0.16132	0.036774
0.30	1.11182	0.86568	0.16812	0.62671	0.38897	0.09922	0.28224	0.21336	0.026206
0.40	1.10424	0.80222	0.12661	0.58390	0.36055	0.09074	0.36090	0.25015	0.018584
0.50	1.08865	0.74199	0.10565	0.54625	0.33604	0.08378	0.43093	0.27467	0.014004
0.60	1.06749	0.68595	0.09331	0.51248	0.31435	0.07784	0.49254	0.28947	0.011348
0.80	1.01441	0.58721	0.07830	0.45361	0.27711	0.06798	0.59196	0.29819	0.008701
1.0	0.95327	0.50499	0.06809	0.40348	0.24581	0.05995	0.66289	0.28924	0.007350
1.5	0.79161	0.35515	0.05017	0.30459	0.18484	0.04473	0.74270	0.23671	0.005333
2.0	0.64031	0.25785	0.03782	0.23190	0.14046	0.03386	0.72912	0.18042	0.004005
3.0	0.40212	0.14417	0.02197	0.13584	0.08214	0.01974	0.58071	0.10110	0.002322
4.0	0.24549	0.08356	0.01291	0.08001	0.04835	0.01161	0.40654	0.05765	0.001363
5.0	0.14778	0.04905	0.007612	0.04723	0.02854	0.006849	0.26635	0.03351	8.035E-4
6.0	0.08828	0.02892	0.004495	0.02791	0.01686	0.004045	0.16798	0.01967	4.745E-4
8.0	0.03117	0.01009	0.001570	0.009751	0.005890	0.001413	0.06285	0.006847	1.657E-4
10.0	0.01093	0.003527	5.488E-4	0.003409	0.002065	4.940E-4	0.02261	0.002392	5.793E-5

Table 4 Functions $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ for $\omega = 0.5$

	a(τ,μ)			b(τ,μ)			c(τ,μ)		
τ	μ=1.0	μ=0.5	μ=0.1	μ=1.0	μ=0.5	μ=0.1	μ=1.0	μ=0.5	μ=0.1
0	1.00000	1.00000	1.00000	0.25126	0.18774	0.072369	0.000000	0.000000	0.000000
0.001	1.00095	0.99995	0.99199	0.24956	0.18616	0.071133	0.001000	0.000999	0.000991
0.002	1.00154	0.99955	0.98371	0.24821	0.18493	0.070246	0.002000	0.001996	0.001964
0.005	1.00272	0.99774	0.95875	0.24477	0.18185	0.068137	0.004996	0.004971	0.004776
0.010	1.00373	0.99380	0.91787	0.23996	0.17763	0.065452	0.009975	0.009876	0.009118
0.020	1.00404	0.98437	0.84036	0.23196	0.17075	0.061429	0.01987	0.01947	0.016605
0.050	0.99907	0.95137	0.64410	0.21354	0.15542	0.053476	0.04889	0.04653	0.031299
0.10	0.98226	0.89194	0.41618	0.19087	0.13725	0.045247	0.09485	0.08594	0.039127
0.20	0.93603	0.77514	0.18379	0.15846	0.11225	0.035297	0.17715	0.14567	0.031062
0.30	0.88333	0.66921	0.09137	0.13478	0.09456	0.028919	0.24684	0.18446	0.019224
0.40	0.82884	0.57602	0.05298	0.11616	0.08092	0.024287	0.30478	0.20722	0.011273
0.50	0.77468	0.49511	0.03570	0.10099	0.06997	0.020710	0.35202	0.21799	0.006796
0.60	0.72196	0.42534	0.02692	0.08835	0.06094	0.017845	0.38968	0.22000	0.004407
0.80	0.62305	0.31404	0.01828	0.06850	0.04693	0.013531	0.44050	0.20733	0.002390
1.0	0.53438	0.23248	0.01363	0.05375	0.03665	0.010452	0.46513	0.18323	0.001626
1.5	0.35777	0.11189	0.007267	0.03028	0.02048	0.005744	0.45274	0.11633	8.275E-4
2.0	0.23575	0.05588	0.004094	0.01753	0.01180	0.003278	0.38784	0.06679	4.615E-4
3.0	0.09958	0.01564	0.001391	0.006128	0.004102	0.001128	0.23591	0.02002	1.556E-4
4.0	0.04113	0.004954	4.947E-4	0.002211	0.001475	4.033E-4	0.12568	0.006019	5.511E-5
5.0	0.01676	0.001697	1.800E-4	8.114E-4	5.403E-4	1.472E-4	0.06218	0.001915	2.001E-5
6.0	0.00676	6.068E-4	6.640E-5	3.010E-4	2.002E-4	5.443E-5	0.02934	6.463E-4	7.373E-6
8.0	0.001079	8.244E-5	9.260E-6	4.226E-5	2.806E-5	7.609E-6	0.005964	8.273E-5	1.027E-6
10.0	1.692E-4	1.160E-5	1.316E-6	6.022E-6	3.996E-6	1.083E-6	0.001122	1.140E-5	1.458E-7

Table 5 Functions $a(\tau,\mu)$, $b(\tau,\mu)$, and $c(\tau,\mu)$ for $\omega = 0.1$

	a(τ,μ)			b(τ,μ)			c(τ,μ)		
τ	μ=1.0	μ=0.5	μ=0.1	μ=1.0	μ=0.5	μ=0.1	μ=1.0	μ=0.5	μ=0.1
0	1.00000	1.00000	1.00000	0.03682	0.02892	0.01238	0.000000	0.000000	0.000000
0.001	0.99937	0.99837	0.99042	0.03648	0.02861	0.01213	0.000999	0.000998	0.000990
0.002	0.99867	0.99668	0.98086	0.03622	0.02837	0.01195	0.001997	0.001993	0.001961
0.005	0.99646	0.99149	0.95264	0.03555	0.02776	0.01152	0.004979	0.004954	0.004760
0.010	0.99259	0.98273	0.90726	0.03462	0.02692	0.01098	0.009914	0.009816	0.009061
0.020	0.98457	0.96511	0.82269	0.03309	0.02558	0.01017	0.01965	0.01926	0.016417
0.050	0.95979	0.91316	0.61323	0.02964	0.02264	0.008575	0.04780	0.04548	0.030503
0.10	0.91831	0.83145	0.37619	0.02555	0.01925	0.006958	0.09126	0.08259	0.037200
0.20	0.83832	0.68776	0.14317	0.02000	0.01482	0.005077	0.16605	0.13606	0.027739
0.30	0.76387	0.56815	0.05613	0.01619	0.01186	0.003934	0.22638	0.16796	0.015624
0.40	0.69527	0.46905	0.02332	0.01335	0.009705	0.003145	0.27417	0.18423	0.007930
0.50	0.63236	0.38712	0.01070	0.01115	0.008051	0.002565	0.31115	0.18939	0.003868
0.60	0.57482	0.31945	0.005665	0.009400	0.006749	0.002121	0.33889	0.18688	0.001891
0.80	0.4744	0.21755	0.002479	0.006809	0.004846	0.001492	0.37198	0.16844	5.358E-4
1.0	0.39107	0.14824	0.001548	0.005024	0.003552	0.001078	0.38250	0.14231	2.286E-4
1.5	0.24051	0.05711	6.859E-4	0.002472	0.001728	5.120E-4	0.35146	0.08024	8.127E-4
2.0	0.14748	0.02226	3.383E-4	0.001273	8.833E-4	2.579E-4	0.28652	0.04032	3.927E-4
3.0	0.05518	0.003585	9.225E-5	3.643E-4	2.504E-4	7.185E-5	0.16015	0.008722	1.055E-5
4.0	0.02057	6.507E-4	2.727E-5	1.106E-4	7.564E-5	2.148E-5	0.07935	0.001738	3.098E-6
5.0	0.007647	1.386E-4	8.439E-6	3.484E-5	2.374E-5	6.695E-6	0.03680	3.450E-4	9.546E-7
6.0	0.002839	3.473E-5	2.690E-6	1.124E-5	7.638E-6	2.144E-6	0.01636	7.231E-5	3.034E-7
8.0	3.899E-4	3.035E-6	2.893E-7	1.229E-6	8.318E-7	2.320E-7	0.002989	4.320E-6	3.251E-8
10.0	5.341E-5	3.230E-7	3.268E-8	1.403E-7	9.454E-8	2.630E-8	5.107E-4	3.806E-7	3.664E-9

Table 6 Functions $\Phi(\tau)$ and $\phi(\tau)$

	Φ(τ)					φ(τ)				
τ	ω=1.00	ω=0.99	ω=0.90	ω=0.50	ω=0.10	ω=1.00	ω=0.99	ω=0.90	ω=0.50	ω=0.10
0	∞	∞	∞	∞	∞	1.00000	0.90909	0.75975	0.58579	0.51317
0.001	4.2483	4.04450	3.41388	1.69895	0.32707	1.00000	0.90863	0.75839	0.58303	0.50964
0.002	3.89375	3.69283	3.09277	1.51911	0.28456	1.00000	0.90824	0.75725	0.58077	0.50680
0.005	3.46806	3.26952	2.70432	1.29988	0.24033	1.00000	0.90720	0.75423	0.57486	0.49942
0.010	3.14363	2.94557	2.40472	1.12906	0.20573	1.00000	0.90564	0.74977	0.56634	0.48893
0.020	2.83696	2.63705	2.11560	0.96178	0.17171	1.00000	0.90284	0.74189	0.55169	0.47122
0.050	2.46537	2.25641	1.74875	0.74456	0.12746	1.00000	0.89552	0.72183	0.51621	0.42962
0.10	2.22200	1.99704	1.48539	0.58456	0.09517	1.00000	0.88487	0.69371	0.46989	0.37776
0.20	2.02240	1.76698	1.23273	0.42970	0.06486	1.00000	0.86601	0.64649	0.39950	0.30371
0.30	1.92968	1.64569	1.08727	0.34260	0.04865	1.00000	0.84883	0.60590	0.34534	0.25050
0.40	1.87555	1.56436	0.98343	0.28315	0.03814	1.00000	0.83264	0.56961	0.30133	0.20964
0.50	1.84034	1.50301	0.90172	0.23889	0.03069	1.00000	0.81716	0.53654	0.26455	0.17718
0.60	1.81594	1.45331	0.83380	0.20428	0.02512	1.00000	0.80223	0.50608	0.23330	0.15082
0.80	1.78509	1.37401	0.72389	0.15331	0.01744	1.00000	0.77371	0.45151	0.18320	0.11102
1.0	1.76720	1.30998	0.63616	0.11765	0.01248	1.00000	0.74662	0.40383	0.14515	0.08296
1.5	1.74621	1.18193	0.47262	0.06405	0.005846	1.00000	0.68383	0.30735	0.08313	0.04177
2.0	1.73836	1.07684	0.35720	0.03636	0.002922	1.00000	0.62686	0.23500	0.04863	0.02185
3.0	1.73352	0.90190	0.20798	0.01245	8.071E-4	1.00000	0.52723	0.13818	0.01722	0.006382
4.0	1.73243	0.75802	0.12225	0.004442	2.402E-4	1.00000	0.44362	0.08152	0.006259	0.001964
5.0	1.73216	0.63767	0.07210	0.001619	7.463E-5	1.00000	0.37331	0.04815	0.002308	6.244E-4
6.0	1.73208	0.53656	0.04258	5.982E-4	2.385E-5	1.00000	0.31415	0.02846	8.591E-4	2.029E-4
8.0	1.73205	0.37997	0.01488	8.353E-5	2.575E-6	1.00000	0.22248	0.009946	1.211E-4	2.240E-5
10.0	1.73205	0.26910	0.005200	1.188E-5	2.914E-7	1.00000	0.15756	0.003477	1.732E-5	2.573E-6

Table 7 Function $q(\tau)$

τ	$q(\tau)$ ω=1.00	ω=0.99	ω=0.90	ω=0.50	ω=0.10
0	0.57735	0.51359	0.41266	0.30174	0.25781
0.001	0.57909	0.51502	0.41350	0.30175	0.25739
0.002	0.58043	0.51608	0.41407	0.30165	0.25696
0.005	0.58375	0.51867	0.41536	0.30118	0.25565
0.010	0.58824	0.52207	0.41683	0.30014	0.25347
0.020	0.59539	0.52727	0.41863	0.29768	0.24915
0.050	0.61076	0.53750	0.42024	0.28925	0.23680
0.10	0.62792	0.54717	0.41790	0.27453	0.21808
0.20	0.64955	0.55550	0.40642	0.24626	0.18626
0.30	0.66337	0.55716	0.39151	0.22067	0.16012
0.40	0.67309	0.55541	0.37539	0.19780	0.13831
0.50	0.68029	0.55163	0.35895	0.17740	0.11991
0.60	0.68580	0.54654	0.34261	0.15922	0.10427
0.80	0.69353	0.53401	0.31106	0.12850	0.07941
1.0	0.69854	0.51975	0.28163	0.10395	0.06093
1.5	0.70513	0.48166	0.21834	0.06168	0.03216
2.0	0.70792	0.44389	0.16857	0.03691	0.01737
3.0	0.70981	0.37484	0.10002	0.01344	0.005288
4.0	0.71027	0.31576	0.05922	0.004963	0.001671
5.0	0.71040	0.26581	0.03504	0.001850	5.413E-4
6.0	0.71043	0.22372	0.02072	6.935E-4	1.783E-4
8.0	0.71044	0.15845	0.007246	9.870E-5	2.008E-5
10.0	0.71045	0.11221	0.002534	1.419E-5	2.337E-6

Table 8 Optical depth at which I_0^+ is maximum, τ_{max}

ω	$\mu_0=1$ $\mu=1.0$	$\mu=0.5$	$\mu=0.1$	$\mu_0=0.5$ $\mu=1.0$	$\mu=0.5$	$\mu=0.1$
1.00	∞	∞	∞	∞	0.20	0.45
0.99	2.6	1.8	1.0	1.8	0.25	0.35
0.90	1.7	1.18	0.48	1.18	0.24	0.45
0.50	1.2	0.80	0.31	0.80	0.21	0.25
0.10	1.05	0.76	0.27	0.76	0.20	0.20

Careful inspection of the expression (34) reveals that $I_0^+(\tau,\mu;\mu_0)$ reaches a maximum value at some depth. Taking the derivative of I_0^+ with respect to τ and using Eq. (36) gives

$$\frac{\partial I_0^+(\tau,\mu;\mu_0)}{\partial\tau}=\frac{\omega I_0\mu_0 H(\mu_0)\,[a(\tau,\mu_0)/\mu_0-a(\tau,\mu)/\mu]}{4\pi(\mu-\mu_0)}$$

When $\mu=\mu_0$, utilization of Eqs. (36) and (37) yields

$$\frac{\partial I_0^+(\tau,\mu;\mu_0)}{\partial\tau}=\omega I_0 H(\mu_0)\frac{[\mu_0 a(\tau,\mu_0)-c(\tau,\mu_0)]}{4\pi\mu_0^2}$$

Thus, $I_0^+(\tau,\mu;\mu_0)$ has a maximum at the optical depth (τ_{max}) where $a(\tau_{max},\mu_0)/\mu_0=a(\tau_{max},\mu)/\mu$ for $\mu\neq\mu_0$ and $\mu_0 a(\tau_{max},\mu_0)=c(\tau_{max},\mu_0)$ for $\mu=\mu_0$. Actually, the maximum for $\mu=\mu_0$ can be determined from the $c(\tau,\mu_0)$ function. Table 8 presents τ_{max} for a given μ_0 and μ. The most pronounced variation in τ_{max} occurs when μ or μ_0 is near unity. The maximum intensity can be found from Eq. (34), i.e., $I_0^+(\tau_{max},\mu;\mu_0)=I_0 H(\mu_0)a(\tau_{max},\mu_0)/4\pi$.

Summary

The intensity distribution inside a semi-infinite, isotropically scattering medium has been expressed in terms of source functions and two universal functions, $a(\tau,\mu)$ and $b(\tau,\mu)$, for various forms of incident radiation. Actually, the universal function $a(\tau,\mu)$ was simply related to the source function for collimated incident radiation. Thus, inspection of the expressions in Eqs. (34, 47, 62, and 72) revealed that the intensity in the positive direction only involved source functions. The universal function, $b(\tau,\mu)$, appeared only in the updwelling intensity expressions, Eqs. (35, 48, 63, and 73). Therefore, the intensity distribution could be found from the source functions and one additional function, $b(\tau,\mu)$. Finally, numerical results were presented for a wide range of parameters.

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References

[1] Kagiwada, H. H. and Kalaba, R. E., "A New Initial-Value Method for Internal Intensities in Radiative Transfer," *Astrophysical Journal*, Vol. 147, Jan. 1967, pp. 301-309.

[2] Kagiwada, H. H. and Kalaba, R. E., "Numerical Results for Internal Intensities in Atmospheres Illuminated by Isotropic Sources," RAND Rept. RM-4958-PR, 1966.

[3] Matsumoto, M., "A New Derivation of a Decomposition Formula," *Journal of Mathematical and Physical Sciences*, Vol. 13, 1969, pp. 393-397.

[4] Ivanov, V. V., *Transfer of Radiation in Spectral Lines*, National Bureau of Standards Spec. Publ. 385, Washington, D.C., 1975.

[5] Chandrasekhar, S., *Radiative Transfer*, Dover Publications, New York, 1960.